Some Co-Axial Magnetic Couplings Designed Using an Analytical Model and an Exact Global Optimization Code

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The purpose of this paper is to use a new rational approach for the design of some magnetic couplings. Our method is based on the association of analytical models and an exact global optimization algorithm named IBBA and developed by the third author. The analytical model presented in this paper is more sophisticated than those previously dealt using IBBA. Therefore, some under and over estimating functions have been constructed for those particular problems of design in order to improve the convergence of IBBA. Some optimal results highlight the efficiency of this approach.

Index Terms—**Analytical model, Branch&Bound, exact global optimization, interval arithmetic, inverse problem, magnetic coupling.**

I. Introduction

THE problem of design is here understood and formulated as an inverse problem. This methodology is well adapted to the design of some electromechanical actuators, see [1], [2]. In order to solve exactly the so-generated global optimization problems, a particular deterministic global algorithm based on a branch and bound technique and on interval arithmetic has been developed by the third author, [3]. In [1] and [2], IBBA was associated with analytical models of permanent magnets motors to find optimized solutions of design.

The associated analytical models need several assumptions to get simplified expressions of electromechanical quantities such as the flux and the torque. Therefore, when optimal solutions of a design problem are obtained, it is often necessary to validate them by means of a numerical model. In order to reduce the use of a numerical model, an analytical model that needs fewer assumptions is proposed. This model is based on the method of separation of the variables for solving Poisson's equations. It gives results that are very close to those obtained by using a finite-element analysis (FEA), [4], [5].

In this paper, IBBA is used once more time but applied to a new application of design concerning magnetic couplings. That is the first time that IBBA is associated with such complicated analytical expressions and thus, some ideas for constructing over and under estimating functions have been introduced. Due to the particular geometrical structure of such devices, it is possible to solve Maxwell equations by separation of the variables. The analytical expressions of electromechanical quantities are more complicated than those employed in [2] but are far simpler than the ones in [4] and [5]. In our knowledge, no other optimization method has been used to solve exactly those kinds of design problems.

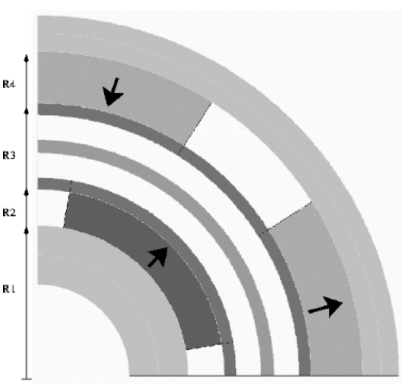

Fig. 1. Geometry of the considered magnetic coupling.

The analytical model is presented in Section II. In Section III, the principle of IBBA is summarized. Its adaptation to our problem is detailed. Section IV presents some magnetic couplings which are optimized using this methodology.

II. Magnetic Coupling

Magnetic coupling are very useful devices for many applications as in chemical and petrochemical industries. In an effort to prevent leakage of hazardous fluids from piping systems, the use of magnetically driven sealless pumps has become more common. Magnetic couplings are used in them to transmit rotational motion without mechanical contact. The studied structure is a co-axial magnetic coupling as shown on Fig. 1. It consists of two rings of permanent magnets separated by a containment can. On each side of it, we find an airgap and a binding band.

A. Parameterization

In order to calculate the torque produced by such a device, one must first choose the parameters required for the computation of the magnetic fields inside it. On Fig. 1, we can see that the two rings of magnet are separated by five concentric areas: two binding bands, two airgap zones and, in the middle, the containment can. These zones are all nonmagnetic, thus we include them into only one big area named *airgap* thereafter. Moreover, we can consider that the two rings of iron yoke have an infinite permeability. Thus, we can take into account only three areas: Area^1 is the inner ring of magnets, Area^3 the outer one and

TABLE I
DEFINITION OF PARAMETERS

Symbol	Quantity	Units
$_{int}$	Angular length of an interior magnet	rad
$_{ext}$	Angular length of an exterior magnet	rad
R_1	Inner radius of interior magnets	m
R_2	Outer radius of interior magnets	m
R_3	Inner radius of exterior magnets	m
R_4	Outer radius of exterior magnets	m
L	Common length of the active magnetic parts	m
p	Number of pole pairs (≥ 2)	-
σ_{mi}	Kind of interior magnet	-
σ_{me}	Kind of exterior magnet	-
σ_{yi}	Kind of inner iron yoke material	-
σ_{ye}	Kind of outer iron yoke material	-

Area^2 the *airgap*. Our structure can be described by 12 parameters (7 reals, 1 integer, 4 categorical variables, see Table I).

B. Modeling

In [4] and [5], an analytical model of the magnetic field in permanent magnet motors is proposed. In [5], a magnetic scalar potential formulation is used. We find the magnetic vector potential formulation described in [4] suitable for magnetic couplings.

1) Magnetic Vector Potential Formulation: As the two armatures of a magnetic coupling rotate at the same speed, the magnetic field can be considered static in a reference fixed on one of them and static Maxwell equations may be applied [4]. The constitutive laws of magnetic media are as follows:

— in the airgap (Area^2)

$$\mathbf{B} = \mu_0 \mathbf{H}. \tag{1}$$

— In the permanent magnets $(\mathrm{Area}^1$ and $\mathrm{Area}^3)$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{J}. \tag{2}$$

From the Gauss's law of magnetism, we can derive the magnetic field from a magnetic vector potential, $\mathbf{A}$. Owing to the geometry of the magnetic coupling (see Fig. 1), we can assume a translational invariance along the Oz axis. Thus, the vector potential has only one non-null component: A_z. The magnetic polarizations of magnets $\mathbf{J}$ are assumed to be purely radial. They have a spatial period equal to $2\pi/p$ in the θ direction. Therefore, in polar coordinates, we can write the harmonic series [4]

$$J_r(r,\theta) = \sum_{n=1}^{\infty} J_{\mathrm{rcn}} \cos(np\theta) + J_{\mathrm{rsn}} \sin(np\theta). \tag{3}$$

The partial differential equation governing A_z is [4]

$$\frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r}\frac{\partial A_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} = \begin{cases} 0, & \text{in Area}^1 \\ \frac{1}{r}\frac{\partial J_r}{\partial \theta}, & \text{otherwise.} \end{cases} \tag{4}$$

To these equations, we must add the boundary conditions and the continuity conditions between two media. As the inner and outer yokes are assumed to have an infinite permeability, we have on surface boundaries at $(r = R_1)$ and $(r = R_4)$:

$$\left(\frac{\partial A_z^1}{\partial r}\right)_{r=R_1} = 0, \quad \left(\frac{\partial A_z^3}{\partial r}\right)_{r=R_4} = 0 \tag{5}$$

where A_z^i is the value of A_z in a media Area^i, A_z^i. Thus, the continuity conditions of the magnetic field must be applied on the interfaces at $(r = R_2)$ and $(r = R_3)$. For instance, on interfaces $(r = R_2)$, we have

$$\left(\frac{\partial A_z^1}{\partial \theta}\right)_{r=R_2} = \left(\frac{\partial A_z^2}{\partial \theta}\right)_{r=R2}, \quad \left(\frac{\partial A_z^1}{\partial r}\right)_{r=R_2} = \left(\frac{\partial A_z^2}{\partial r}\right)_{r=R_2}. \tag{6}$$

Because $(p \geq 2)$, the method of separation of the variables gives the form of the analytical expression of A_z in $Area^i$:

$$A_z^i(r,\theta) = \sum_{n=1}^{\infty} \left[\left(A_n^i r^{np} + B_n^i r^{-np} + \frac{sc_n^i r}{1-(np)^2} \right) \cos(np\theta) + \left(C_n^i r^{np} + D_n^i r^{-np} + \frac{ss_n^i r}{1-(np)^2} \right) \sin(np\theta) \right]. \tag{7}$$

The sc_n^i and ss_n^i coefficients are null in Area^2. In Area^1 and Area^3, we have the following relations:

$$\begin{aligned} sc_n^1 &= 0 \\ ss_n^1 &= -\frac{2pJ_r(\sigma_{\mathrm{int}})}{\pi} \sin\left(\frac{np\theta_{\mathrm{int}}}{2}\right)(1-(-1)^n) \\ sc_n^3 &= -\frac{pJ_r(\sigma_{\mathrm{ext}})}{\pi} \left[\cos\left(\frac{np\theta_{\mathrm{ext}}}{2} + np\theta_0\right) - \cos\left(-\frac{np\theta_{\mathrm{ext}}}{2} + np\theta_0\right) \right](1-(-1)^n) \\ ss_n^3 &= -\frac{pJ_r(\sigma_{\mathrm{ext}})}{\pi} \left[\sin\left(\frac{np\theta_{\mathrm{ext}}}{2} + np\theta_0\right) - \sin\left(-\frac{np\theta_{\mathrm{ext}}}{2} + np\theta_0\right) \right](1-(-1)^n) \end{aligned} \tag{8}$$

where θ_0 is the angular gap between the two armatures and $J_r(\sigma_{\mathrm{int}})$ and $J_r(\sigma_{\mathrm{ext}})$ are respectively the radial polarization of inner and outer magnets. The value of the torque is maximum when $\theta_0 = \pi/2$ (see Fig. 1). Thanks to relations (5) and (6) the values of A_n^i, B_n^i, C_n^i, and D_n^i coefficients appearing in relation (7) can be known by solving the two equations

$$\mathbf{Mat_R}.\{AB\} = \{S_{AB}\} \quad \mathbf{Mat_R}.\{CD\} = \{S_{CD}\} \tag{9}$$

where

$$\begin{aligned} \{AB\}^T &= \{A_n^1, B_n^1, A_n^2, B_n^2, A_n^3, B_n^3\} \\ \{CD\}^T &= \{C_n^1, D_n^1, C_n^2, D_n^2, C_n^3, D_n^3\} \\ \{S_{AB}\}^T &= k_{np}\{0, 0, 0, npR_3 sc_n^3, R_3 sc_n^3, R_4 sc_n^3\} \\ \{S_{CD}\}^T &= k_{np}\{R_1 ss_n^1, npR_2 ss_n^1, R_2 ss_n^1, npR_3 ss_n^3, \\ &\qquad R_3 ss_n^3, R_4 ss_n^3\} \\ k_{np} &= \frac{1}{np(1-(np)^2)} \end{aligned}$$

$$\mathbf{Mat_R} = \begin{pmatrix} -R_1^{np} & R_1^{-np} & 0 & 0 & 0 & 0 \\ -R_2^{np} & -R_2^{-np} & R_2^{np} & R_2^{-np} & 0 & 0 \\ -R_2^{np} & R_2^{-np} & R_2^{np} & -R_2^{-np} & 0 & 0 \\ 0 & 0 & R_3^{np} & R_3^{-np} & -R_3^{np} & -R_3^{-np} \\ 0 & 0 & R_3^{np} & -R_3^{-np} & -R_3^{np} & R_3^{-np} \\ 0 & 0 & 0 & 0 & -R_4^{np} & R_4^{-np} \end{pmatrix}.$$

2) Magnetic Torque: The magnetic flux density $\mathbf{B^i}$ in each Area^i can be calculated from the magnetic vector potential Az.

Hence, by applying the Maxwell stress tensor in the airgap, the torque $\Gamma_{\rm em}$ can be obtained. Comparisons with FEA show that we can take into account only the first harmonic $(n=1)$. Thus, using MAPLE, we can get an analytical expression of the torque (10)

$$\begin{aligned}\Gamma_{\rm em} = &\frac{-8pL}{\mu_0\pi(p^2-1)^2\left(R_4^{2p}-R_1^{2p}\right)}\\ &\times J_r(\sigma_{mi})\sin(p\theta_{\rm int}/2)J_r(\sigma_{me})\sin(p\theta_{\rm ext}/2)\\ &\times\Big((p-1)R_2^{p+1}+2R_1^{p+1}-(p+1)R_1^{2p}/R_2^{p-1}\Big)\\ &\times\Big((p-1)R_3^{p+1}+2R_4^{p+1}-(p+1)R_4^{2p}/R_3^{p-1}\Big). \quad (10)\end{aligned}$$

3) Thickness of Iron Yokes: We present here a simple method which allows to calculate an approximation of the thickness of each iron yoke (t_y). Indeed, the mean value of magnetic flux density B_θ^y in the yokes must be less or equal to the maximum value $B_M(\sigma_y)$ above which the iron is definitely saturated.

In the yoke, the maximum value of magnetic flux is $\Phi_y = B_\theta^y t_y L$ and the magnetic flux per pole is given by

$$\Phi_m = \iint \mathbf{B}.\mathbf{dS} = \int_\theta \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta}\right) Lrd\theta = 2L.\mathrm{Max}(A_z) \quad (11)$$

where $\mathrm{Max}(Az)$ is the maximum value of the magnetic vector potential and can be obtained from relations (7) and (9). According to the Gauss's law of magnetism we have $\Phi_m = 2\Phi_y$. Eventually, the analytical expression of the thickness of the inner yoke t_y^1 is

$$\begin{aligned}t_y^1 = &\frac{4J_r(\sigma_{mi})\sin(p\theta_{\rm int}/2)}{\pi B_M(\sigma_{yi})(p^2-1)\left(R_4^{2p}-R_1^{2p}\right)}\Bigg[(p-1)R_1^{2p}\\ &\left(R_2^{p+1}-R_1^{p+1}\right)+(p+1)R_4^{2p}R_1\left(1-\left(\frac{R_1}{R_2}\right)^{p-1}\right)\Bigg]\end{aligned}$$

where $B_M(\sigma_{yi})$ is the maximum flux density in the inner iron yoke. An analogue expression for the thickness of the outer yoke t_y^3 can be derived.

III. Formalization of the Design Problem

A. Formulation

An inverse problem is formulated as mixed constrained global optimization problems defined as follows:

$$\begin{cases}\min\limits_{\substack{x\in R^n,z\in N^m\\ \sigma\in\prod_{i=1}^{l}K^i,b\in B^r}} f(x,z,\sigma,b)\\ \text{subjected to:}\\ \quad g_i(x,z,\sigma,b)\le 0\ \forall i\in\{1,\ldots,p\}\\ \quad h_j(x,z,\sigma,b)=0\ \forall j\in\{1,\ldots,q\}\end{cases} \quad (12)$$

where f is a real function; K^i represents an enumerated set of categorical variables, that is for example the type of magnet; and $B=\{0,1\}$ the boolean set which is used to model, for example, the fact that an actuator is with or without slot(s).

B. IBBA Algorithm

Interval analysis was introduced by Moore in 1966 [6]. All real values are enclosed by an interval where the bounds

TABLE II
LIST OF CONSTRAINTS

Number	Relation
C_1	$\Gamma_{em} = \Gamma a_{fixed}$
C_2	$R_2 - R_1 \ge t_{yi}^{min}$
C_3	$R_3 - R_2 \ge g^{min}$
C_4	$R_3 - R_2 \le g^{max}$
C_5	$R_4 - R_3 \ge t_{ye}^{min}$
C_6	$R_4 \ge \lambda^l L$
C_7	$R_4 \le \lambda^u L$
C_8	$R_1 - t_y^1 \ge R_{int}^{min}$

are the two closest floating point numbers. Expanding the classical operations—addition, subtraction, multiplication and division—into intervals, defines interval arithmetic. A straightforward generalization allows computations of reliable bounds of a function over a hypercube (or box) defined by an interval vector. Moreover, classical tools of analysis such as Taylor expansions can be used together with interval arithmetic to compute more precise bounds [6], [7].

The principle of IBBA is first to bisect the initial domain into smaller and smaller boxes and then, to eliminate the boxes where the global optimum cannot occurs:

- by proving, using interval bounds, that no point in a box can produce a better solution than the current best one;
- by proving, with interval arithmetic, that at least one constraint cannot be satisfied by any point in such a box.

To accelerate the convergence, constraint propagation techniques are used in some steps of IBBA [8]. Such interval Branch and Bound algorithms guarantee to produce an ε-global optimal solution, where $\varepsilon > 0$ is the maximal error on the objective function value; ε is fixed by the user of IBBA. Actually, IBBA was used to solved exactly some non-homogeneous mixed constrained global optimization problem with almost 25 variables and 15 parameters (taking between few seconds to 6 days of computing times). The difficulty to solve a problem depends on the number of variables and of constraints but also on the complexity of the analytical equations. Indeed interval arithmetic has not all the properties of the classical one (for example it is sub-distributive). Therefore, bounds computed using interval arithmetic can be very large [6]. This is the main difficulty that we have to deal with in this work. For details on such an algorithm see [3] and [7].

C. Application and Adaptations of IBBA

Our input vector is defined by the parameters used in Table I. The minimum value of the volume of magnets $\mathbf{V_a}$ or the global volume $\mathbf{V_g}$ is searched

$$\mathbf{V_a} = pL\left(\theta_{\rm int}\left(R_2^2-R_1^2\right)+\theta_{ext}\left(R_4^2-R_3^2\right)\right. \quad (13)$$

$$\mathbf{V_g} = pL\left[\left(R_4+t_y^3\right)^2-\left(R_1-t_y^1\right)^2\right]. \quad (14)$$

Each solution must satisfy an equality constraint upon the fixed electromagnetic torque Γ_{em} (10), and seven geometric inequalities (see Table II), $\Gamma_{\rm fixed}$ is the value of torque imposed by the schedule of conditions; $t_{yi}^{\rm min}$, $g^{\rm min}$, $t_{ye}^{\rm min}$, and $R_{\rm int}^{\rm min}$ are respectively the minimum values of the thicknesses of the interior magnets, the airgap, the exterior magnets and the inner radius of our device; $g^{\rm max}$ is the maximal value of the airgap; $\lambda = [\lambda^l, \lambda^u]$ is a shape coefficient.

TABLE III
RESULTS

Parameters	Bounds	Unit	Min $\mathbf{V_a}$	Min $\mathbf{V_g}$
θ_{int}	[0.001;1] π/p	rad	38.15	48.65
θ_{int}	[0.001;1] π/p	rad	25.75	37.80
R_1	[10;200]	mm	26.785	22.015
R_2	[10;200]	mm	29.785	25.015
R_3	[10;200]	mm	31.785	27.015
R_4	[10;200]	mm	34.785	30.015
L	[50;500]	mm	50.22	50.025
p	[2,7]	-	7	4
σ_{ai}	[1,2]	-	2	2
σ_{ae}	[1,2]	-	2	2
σ_{yi}	[1,2]	-	1	2
σ_{ye}	[1,2]	-	1	2
t_y^1		mm	1.177	1.724
t_y^3		mm	0.867	1.448
V_a		cm^3	**18.33**	20.95
V_g		cm^3	97.08	**90.87**
Γ_{em}		N.m	9.806	9.809
CPU Time		hours	161	117
Iterations		-	2210055	2035197

In front of the complexity and the strong nonlinearity of many expressions (criteria, torque, thicknesses of yoke), the use of interval arithmetic is not easy. The returned bounds are not efficient enough. Hence, to improve the convergence of IBBA, we propose to intersect Γ_{em} with an undervaluation Γ_{em}^{inf} and an overvaluation of it Γ_{em}^{sup}. We use for Γ_{em}^{inf}

$$\Gamma_{em}^{\text{inf}} = \frac{2.10^7 p^2 L \left(R_4^{p+1} - R_3^{p+1}\right)\left(R_2^{p+1} - R_1^{p+1}\right)}{\pi^2 (p+1)^2 \left(R_4^{2p} - R_1^{2p}\right)} \times J_r(\sigma_{mi}) \sin(p\theta_{\text{int}}/2) J_r(\sigma_{me}) \sin(p\theta_{\text{ext}}/2). \quad (15)$$

This procedure replaces the constraint C_1. An example can explain it. If, during the process, our one-column input vector $\{\mathbf{X}\}$ is:

$$\{\mathbf{X}\}^T = \left\{ \begin{array}{c} [0.637, 0.649] \times \pi/5, [0.662, 0.675] \times \pi/5, \\ [0.297, 0.320], \\ [0.300, 0.324], [0.373, 0.397], [0.376, 0.400], \\ [0.466, 0.480], [5, 5], [1, 1], [2, 2], [2, 2], [2, 2] \end{array} \right\}.$$

Therefore, the obtained intervals of the torque without and with these limiting actions are, respectively:

$$\Gamma_{em} = [-55513, 50604]$$
$$\Gamma_{em} \bigcap \left[\min\left(\Gamma_{em}^{\text{inf}}\right), \max\left(\Gamma_{em}^{\text{sup}}\right)\right] = [-6589, 1064].$$

A same method has been applied to t_y^1 and t_y^3, [9].

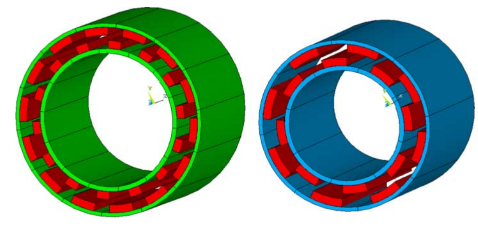

Fig. 2. Geometries of the two optimal solution (Min $\mathbf{V_a}$ and Min $\mathbf{V_g}$).

IV. RESULTS AND CONCLUSION

To illustrate the use of this new IBBA algorithm, let us consider the following optimal design problem. It deals with the minimization of our two criteria ($\mathbf{V_a}$ and $\mathbf{V_g}$). The equality constraint is true when the torque is equal to 10 ± 0.2 N·m. The magnets can be in Sm2Co17 (if σ_{mi} or $\sigma_{me} = 1$) or in NdFeB (if σ_{mi} or $\sigma_{me} = 2$). The iron yokes can be in steel z15 (if σ_{yi} or $\sigma_{ye} = 1$) or in z30c13 (if σ_{yi} or $\sigma_{ye} = 2$). The initial bounds of our parameters and the obtained results are shown in Table III and on Fig. 2. First we can see that the value of torque is close to the allowed lower limit (9.8 N·m). We can also note that the obtained numbers of pole pairs are clearly different (4 and 7). Moreover the kinds of steel of yokes are different.

These results show the efficiency of our approach. Indeed without the dimensions, the integer and the categorical parameters describe a set of 116 (7×2^4) different geometries of magnetic couplings. The method is able to find the most efficient one and the result is guaranteed.

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