Hybrid Analytical Model Coupling Laplace's Equation and Reluctance Network for Electrical Machines

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***Abstract*-This paper presents a hybrid magnetic model combining reluctance network and Laplace's equation resolution using magnetic vector potential in the air gap. The coupling of the two models is performed by introducing fictive equivalent magnetomotive forces in the air-gap. The model is applied to the study of DC machines used as a starter in automotive application. Good results are obtained compared to finite element model and experimental measurements.**

Index Terms—**hybrid magnetic model, analytical formulation, reluctance network.**

I. Introduction

Analytical models based on the resolution of the Laplace equation using the method of separation of variables have been developed for the modeling of electrical machines as in [1]. These models are accurate and fast but do not take into account the nonlinear aspects of ferromagnetic materials. To improve this method, the saturation of the iron yoke can be included by making a hybrid model between finite elements (FE)in the iron parts and Laplace equation in the air-gap as presented in [2]. This FE hybrid model showed its effectiveness through its use in a geometrical optimization process in [3]. However, in some cases, we need to model the complete geometry of the studied device, so the FE part of the model become time consuming. An alternate way consists in introducing reluctance network (RN) models in ferromagnetic parts of electrical machines instead of modeling them by FE method [4,5]. In these RN hybrid model the air-gap is modeled by a set of reluctances calculated from the analytical expression of the magnetic potential vector (MVP). This kind of RN hybrid model makes the model complex to develop. Indeed, each air-gap terminal of the boundary stator or rotor reluctance is connected to the whole other air-gap terminals, this method needs a high number of air-gap reluctances to take into account the rotation of the machine. The coupling between the stator and rotor is then made by circuit connections.

In this paper, the authors present an alternate method of coupling the reluctances of the stator and the rotor with a set of magnetomotive forces (MMFs) which depend on these reluctances fluxes. With this method and thanks to the expression of these MMFs depending on all stator and rotor boundaries reluctance fluxes, the number of circuit connections is reduced to one reference point. This point is arbitrary chosen in the air-gap, to link the different MMFs.

This hybrid model reduces the number of air-gap connections and simplifies the model making it faster. The developed model is applied on a DC machine. Based on this example, the principle of the magnetomotive forces coupling is validated thanks to corresponding finite element models (FEM) and measurements.

II. Airgap modeling principle

The principle of the hybrid model is applied to a DC machine in two dimensions, as shown in Fig. 1. In this model, the air and the iron yoke of the rotor and stator are modeled by a reluctance network, while the air gap is modeled by a set of magnetomotive forces F_k. These air-gap MMFs are calculated from the terminal P_k of the reluctance R_k to the reference point P_0. Indeed, this reference point P_0 is a zero point potential arbitrarily chosen in the airgap. In the following, we will show how these MMFs are calculated from the resolution of Laplace equation.

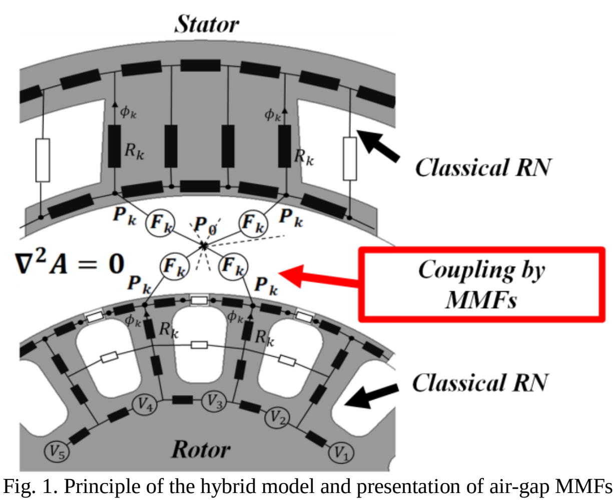

Fig. 1. Principle of the hybrid model and presentation of air-gap MMFs

A. Modeling of air-gap magnetomotive forces

The magnetomotive force is a magnetic potential difference calculated between two points from P_k to P_0, it is therefore expressed using the magnetic field $\boldsymbol{H}$:

$$F = \int_{P_k}^{P_0} \boldsymbol{H} \cdot d\boldsymbol{l} = F_r + F_\theta \tag{1}$$

The magnetomotive force can be expressed as a sum of two terms depending on the radial and the tangential components of the magnetic field (as shown in Fig. 2):

$$\begin{cases} F_r = \int_{r_k}^{r_i} H_r\, dr = \frac{1}{\mu_0}\int_{r_k}^{r_i} \frac{1}{r}\frac{\partial A_z(r,\theta)}{\partial \theta} dr \\ F_\theta = \int_{\theta_i}^{\theta_0} H_\theta\, r_0 d\theta = -\frac{1}{\mu_0}\int_{\theta_i}^{\theta_0} \frac{\partial A_z(r,\theta)}{\partial r} r_0 d\theta \end{cases} \quad (2)$$

where A_z is the axial component of the magnetic vector potential (MVP).

In this model, the coupling between the stator and the rotor is not introduced by circuit connections as shown in Fig. 1 but it lies in the global expressions of the MMFs. Indeed, each MMF depends on the MVP as in (2). The calculation of the MVP by the variable separation method needs to use the boundaries conditions on both stator and rotor radii r_{si} and r_{re}. These boundaries conditions use the fluxes ϕ_S and ϕ_R provided by the reluctance networks of the stator respectively.

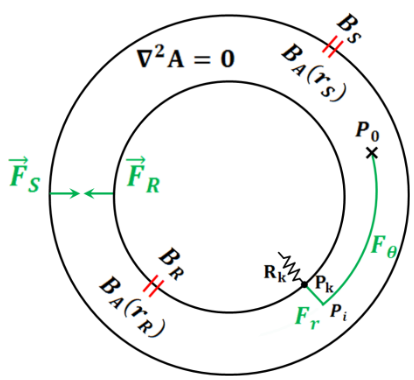

Fig. 2. MMFs calculation principle between each reluctance in the boundary of the air-gap and the reference point

The MMFs $[\boldsymbol{F_S}]$ and $[\boldsymbol{F_R}]$,respectively connected to the stator and the rotor, can be expressed in terms of vector of fluxes $[\boldsymbol{\phi_S}]$ and $[\boldsymbol{\phi_R}]$ as follows:

$$\begin{bmatrix}[\boldsymbol{F_S}]\\ [\boldsymbol{F_R}]\end{bmatrix} = \begin{bmatrix}[\boldsymbol{K_{SS}}][\boldsymbol{K_{SR}}]\\ [\boldsymbol{K_{RS}}][\boldsymbol{K_{RR}}]\end{bmatrix}\begin{bmatrix}[\boldsymbol{\phi_S}]\\ [\boldsymbol{\phi_R}]\end{bmatrix} \quad (3)$$

$[\boldsymbol{K_{SS}}]$, $[\boldsymbol{K_{SR}}]$, $[\boldsymbol{K_{RS}}]$, $[\boldsymbol{K_{RR}}]$ are matrices determined by using the expression of the MVP in the air-gap. These matrices have the same unit than reluctances, but do not correspond to fixed flux tubes. The above matrices are determined using the definition of the MMFs (2) and the resolution of the Laplace equation in the air-gap which is developed in the following.

B. Laplace equation resolution

The air-gap MVP A_z is determined by solving Laplace equation (in polar coordinates) using the method of separation of variables, as explained in [6]. The general expression of the solution is given by:

$$A_z(r,\theta) = (a_0 + b_0 \ln(r)) + \sum_{n=1}^{N_h} \begin{matrix}(\alpha_{1n} r^n + \alpha_{2n} r^{-n})\cos(np\theta) \\ +(\beta_{1n} r^n + \beta_{2n} r^{-n})\sin(np\theta)\end{matrix} \quad (4)$$

where N_h is the number of harmonics considered and p means the periodicity coefficient. Both a_0 and b_0 are zero due to the periodicity of the solution. $\alpha_{1n}, \alpha_{2n}, \beta_{1n}, \beta_{2n}$ are unknown coefficients that can be determined using the boundary conditions on $r = r_{si}$ and $r = r_{re}$. On these boundaries, the normal flux density B_r is considered using the fluxes ϕ_S and ϕ_R provided by the reluctance network. It is assumed that the tangential flux density B_θ is zero at both boundaries.

C. Boundaries conditions

From each terminal P_k belonging to the air-gap boundaries connected to a reluctance R_k, flows a flux ϕ_k. According to RN principle, the flux density corresponding to each reluctance is constant over the cross section of the flux tube.

$$B_{r,k}(\theta) = \frac{\phi_k}{S_k} \nu_k(\theta) \quad (5)$$

where:

$B_{r,k}$: corresponds to the normal flux density due to the stator or rotor reluctances.

ν_k: is the support function of the flux density created by the reluctance R_k, noticed ν_k^S and ν_k^R when defined to the stator and rotor boundaries respectively.

S_k: is the cross-section surface of the corresponding flux tube, noticed S_k^S and S_k^R when defined in the stator and rotor boundaries respectively.

Using the superposition principle in the air-gap, the total flux density on the boundaries $r = r_{si}$ and $r = r_{re}$ is expressed as the sum of elementary flux densities created by the fluxes of the different reluctances such that:

$$B_r(r_{si},\theta) = \sum_{k=1}^{N_S} \frac{\phi_k^S}{S_k^S} \nu_k^S(\theta) \;\; ; \;\; B_r(r_{re},\theta) = \sum_{k=1}^{N_R} \frac{\phi_k^R}{S_k^R} \nu_k^R(\theta) \quad (6)$$

N_S and N_R are respectively the numbers of the reluctances in the stator and the rotor boundaries. ϕ_k^S and ϕ_k^R are the elementary fluxes of the vectors $[\phi_S]$ and $[\phi_R]$ used in (3).

To find the unknown coefficients of the MVP according to (4), the general support function is expressed by its spectral decomposition:

$$\nu_k(\theta) = \frac{w_k}{2\pi} + \sum_{n=1}^{N_h} C_n(k)\cos(np\theta) + D_n(k)\sin(np\theta) \quad (7)$$

with:

$$\begin{cases} C_n(k) = \frac{2}{2\pi n}\left(\sin\left(n\left(\theta_k + \theta + \frac{w_k}{2}\right)\right) - \sin\left(n\left(\theta_k + \theta - \frac{w_k}{2}\right)\right)\right) \\ D_n(k) = \frac{2}{2\pi n}\left(\cos\left(n\left(\theta_k + \theta - \frac{w_k}{2}\right)\right) - \cos\left(n\left(\theta_k + \theta + \frac{w_k}{2}\right)\right)\right) \end{cases} \quad (8)$$

where:

θ_k: is the position of the corresponding reluctance.

w_k: is the width of the flux tube.

The boundaries conditions are expressed as following:

$$\begin{cases} B_r(r_{re},\theta) = (1/r \cdot \partial_\theta A_z(r,\theta))_{r_{re}} \\ B_r(r_{si},\theta) = (1/r \cdot \partial_\theta A_z(r,\theta))_{r_{si}} \end{cases} \quad (9)$$

From (4), the normal flux density can be expressed as:

$$\frac{1}{r}\partial_\theta A_z(r,\theta) = \sum_{n=1}^{N_h} -np(\alpha_{1n} r^{n-1} + \alpha_{2n} r^{-n-1})\sin(np\theta) + np(\beta_{1n} r^{n-1} + \beta_{2n} r^{-n-1})\cos(np\theta) \quad (10)$$

Combining the equations (5), (7) and (10) the four coefficients $(\alpha_{1n}, \alpha_{2n}, \beta_{1n}, \beta_{2n})$ are determined for each harmonic and gathered in the vectors:

$$\begin{cases} [\alpha_1] = [L_1][\phi_R] + [L_2][\phi_S] \\ [\alpha_2] = [M_1][\phi_R] + [M_2][\phi_S] \\ [\beta_1] = [N_1][\phi_R] + [N_2][\phi_S] \\ [\beta_2] = [O_1][\phi_R] + [O_2][\phi_S] \end{cases} \quad (11)$$

where $[L_1], [L_2], [M_1], [M_2], [N_1], [N_2], [O_1]$ and $[O_2]$ are matrices depending on the geometrical parameters of the considered machine.
According to (11) the airgap MVP $A_z(r, \theta)$ is determined with the fluxes $[\phi_R]$ and $[\phi_S]$ provided by the reluctance networks. In the following, the calculation of MMFs is detailed.

D. *Magnetomotive forces calculation*

According to (2) and (4), the MMFs F_r and F_θ can be expressed as:

$$F_{r,k} = \frac{1}{\mu_0} \sum_{n=1}^{N_h} -p \begin{pmatrix} \alpha_{1n}(r_i^n - r_k^n) \\ -\alpha_{2n}(r_i^{-n} - r_k^{-n}) \end{pmatrix} \sin(np\theta_k) + p \begin{pmatrix} \beta_{1n}(r_i^n - r_k^n) \\ -\beta_{2n}(r_i^{-n} - r_k^{-n}) \end{pmatrix} \cos(np\theta_k) \quad (12)$$

$$F_{\theta,k} = -\frac{1}{\mu_0} \sum_{n=1}^{N_h} \frac{1}{p} . \begin{pmatrix} \alpha_{1n} r_0^n \\ -\alpha_{2n} r_0^{-n} \end{pmatrix} \begin{pmatrix} sin(np\theta_0) \\ -\, sin(np\theta_i) \end{pmatrix} - \frac{1}{p} . \begin{pmatrix} \beta_{1n} r_0^n \\ -\beta_{2n} r_0^{-n} \end{pmatrix} \begin{pmatrix} \cos(np\theta_0) \\ -\cos(np\theta_i) \end{pmatrix} \quad (13)$$

Introducing (11) in the two previous equations, normal and tangential magnetomotive forces are expressed in terms of rotor and stator fluxes ϕ_S and ϕ_R:

$$F = f(\phi_R, \phi_S) \quad (14)$$

Finally using the definition (3), the expressions of the matrices $[K_{SS}], [K_{SR}], [K_{RS}], [K_{RR}]$are identified.

As explained in section II.A the coupling method developed in this paper does not depend on any circuit connection but lies on the global expression of the air-gap MMFs. This approach allows to avoid the complexity and the drawbacks of the classical reluctance networks.
Thanks to this model, the rotor movement does not introduce any problem in the simulation of the machine and the torque waveforms can be easily determined.

E. *Torque calculation by Maxwell stress tensor*

Usually in reluctance networks, the torque is calculated with virtual works principle using the derivation of the coenergy. In the developed hybrid model, normal and tangential components of the air-gap flux densityare known through MVP A_z. Therefore, Maxwell stress tensor can be used on a circular path of radius r_m in the air-gap to evaluate the electromagnetic torque :

$$T_{em} = \frac{r_m^2 \, L_z}{\mu_0} \int_0^{2\pi} B_\theta(r_m, \theta) \, B_r \, (r_m, \theta) \, d\theta \quad (15)$$

with L_z the stack length of the machine.

The normal and tangential components of the flux density are expressed in terms of magnetic vector potential which coefficients $\alpha_{1n}, \alpha_{2n}, \beta_{1n}$ and β_{2n} are depending on the fluxes $[\phi_R]$ and $[\phi_S]$. The use of separation of variables method enables to express all the quantities in terms of their spectral decomposition coefficient. By replacing the integral $\int_0^{2\pi} B_\theta B_r \; d\theta$, the final expression of the torque is then simplified to:

$$T_{em} = \frac{2\pi L_z}{\mu_0} \sum_{n=0}^{N_h} -n^2 p \, (\beta_{2n}\alpha_{1n} - \beta_{1n}\alpha_{2n}) \quad (16)$$

III. VALIDATION OF THE MODEL

The model is validated by comparing the obtained magnetic characteristics to the corresponding 2D FEM. The modeled machine is a 4-pole 19-slot DC series motor used in automotive starters. In this application the machine is subjected to very high saturation levels far exceeding 2T in a magnetic material with a saturation magnetization 2T as shown in Fig. 3. The geometrical parameters of the machine are listed in TABLE I.

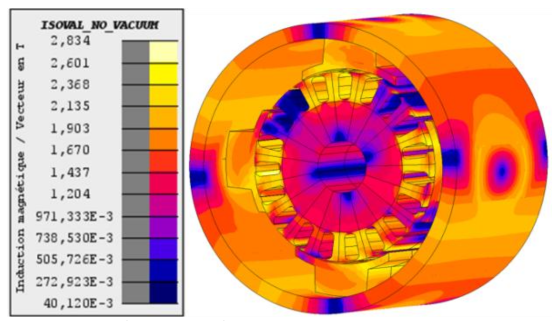

Fig. 3. 3D FEM and field map of the DC Machine

TABLE I
GEOMETRICAL PARAMETERS OF THE MACHINE

Rotor outer radius → 24,85 10^{-3} m	Rotor stack length → 35 10^{-3} m
Air-gap thickness → 0,65 10^{-3} m	Poles axial length → 30,5 10^{-3} m
Machine outer radius → 38 10^{-3} m	Stator yoke length → 60 10^{-3} m

The 2D FE mesh of the machine is shown in Fig. 4, where the yellow parts represent windings, the white parts vacuum, and the gray parts the stator and rotor iron core.

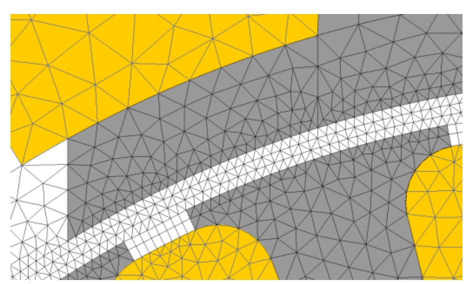
Fig. 4. Mesh of the machine in the corresponding 2D FE model.

The proposed hybrid model is tested under several conditions in order to validate the accuracy of the MMFs coupling method. Firstly, the flux density waveforms under the stator poles of the DC series machine given by full FEM and hybrid model is plotted in Fig. 5. It can be seen on this figure, that the blue curve provided by the hybrid model and the red curve given by FEM have the same shape. In fact, the hybrid model is based on Fourier series with 100 harmonics while the FEM is based on an air-gap discretization with 730 element over 2π.

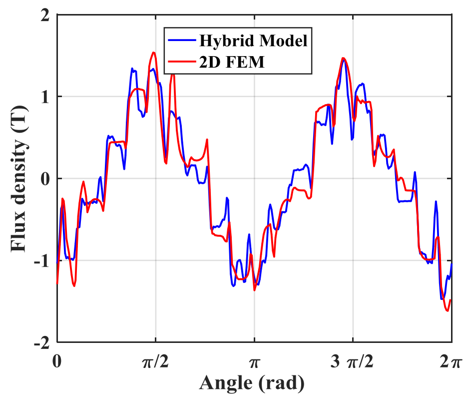

Fig. 5. Air-gap flux density waveforms over one period

Moreover, for a given rotor position, the torque-current curves are calculated for a wide range of the supply current. The high value of the current are characterized by a very high level of saturation and armature reaction. The comparison is performed between the presented hybrid model, 2D and 3D FEM and experimental measurements. The 4 torque-current curves are plotted on Fig. 6 in p.u. We note that the hybrid model and 2D FEM give very similar results which proves the accuracy of the model for a wide range of the current.

There is a very slight difference between the results of 2D and 3D FEM due to the consideration of the 3D effects by correcting factor in the 2D FEM, as described in [7]. The agreement with experimental results validates all the models, and especially the proposed hybrid-model.

Regarding the CPU time, using an Intel Xeon @ 2.4 GHz the hybrid model takes ~2.5 seconds, while the 2D FEM takes ~36 seconds and 3D FEM takes ~8 minutes for each operation point.

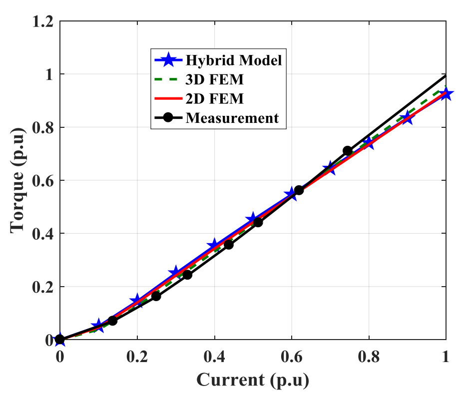

Fig. 6. Torque evolution for a given rotor position and several current values.

Considering the rotor position in the MMFs calculation with field model, the torque is calculated for several rotor positions over one pole pitch (Fig. 7) with both hybrid and 2D FEM. Once again, a really good agreement between the two models is observed. This is not the case for classical reluctance network models.

Then, we can highlight the advantages of the proposed hybrid model:

- Same accuracy and precision as 2D or 3D FEM.
- Low CPU time thanks to reduction of air-gap elements.
- Easier implementation than classical RN models accounting for the rotation (using only one air-gap reference point P_0).

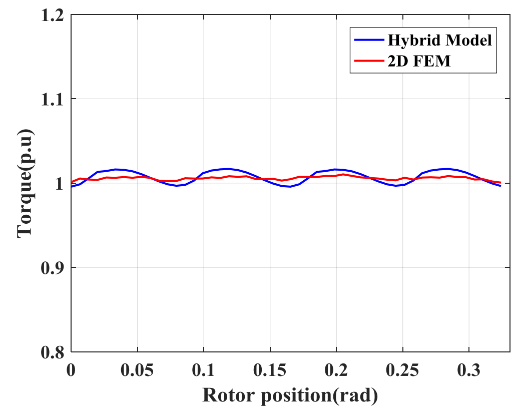

Fig. 7. Torque-position curve in highly saturated conditions

IV. Conclusion

In this paper a hybrid model based on the coupling between reluctance network and the resolution of Laplace equation is developed and tested on a DC machine. The accuracy of the coupling using air-gap MMFs is validated by an experimental result as well as by 2D and 3D FEM. This type of models allows both simplifying the rotation modeling and, at the same time makes the model robust using the method of MMF coupling. The accuracy of the model and its fastness make it of a high interest to be used in an optimization process. Indeed, the coupling by MMFs allows us to have a very high robust model adaptable to geometry changes in a wide range of variation.

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